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Counting digraphs and hypergraphs

Colm Ó Dúnlaing

Chee K. Yap

Courant Institute of Mathematical Sciences

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ABSTRACT

Formulae are derived for the number of digraphs and of hypergraphs with given automorphism group. It is shown that for $n > 2$ (respectively, $n > 3$), there is no hypergraph (respectively, digraph) with automorphism group A_n.

Counting digraphs and hypergraphs

Colm Ó'Dúnlaing
Chee K. Yap
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, New York 10012

In this note we show some combinatorial results about automorphism groups of digraphs and hypergraphs. The problems arose in connection with some issues in database theory: in [HY] the notion of *data base formats* is introduced. It is shown [OY] that a natural concept of 'information capacity' of formats is closely connected to the automorphism groups of instances of formats. Digraphs (resp. hypergraphs) turn out to be the instances of a very simple format.

For any positive integer n, S_n is the permutation group on $\{1,\ldots, n\}$, i.e., the group of all $n!$ bijective maps from this set onto itself. A *digraph* (directed graph) on this set of vertices may be identified with a set of ordered pairs of distinct vertices, i.e., a digraph may be uniquely defined by specifying its directed edges. Again, a *hypergraph* on this set of vertices may be identified as a set of subsets of the set of vertices, each subset being a 'hyper-edge.' Notice that a graph can be regarded as a hypergraph in which each hyperedge has exactly two vertices.

Given a hypergraph H, if π is a permutation of the vertices such that for every hyperedge E in H, $\pi E = \{\pi v \colon v \in E\}$ also belongs to H, then we call π an *automorphism* of H. Clearly the set of automorphisms of H forms a subgroup of S_n. Similarly one defines the automorphism group of a digraph D, based on the definition $\pi(u,v) = (\pi u, \pi v)$ for a directed edge (u,v).

A *cycle* or cyclic permutation, written $(ij...m)$ where i, j,.. etcetera are distinct numbers from $\{1,\ldots, n\}$, is one that maps i to j, j to k, ... and m to i, leaving the other numbers fixed. Clearly, two disjoint cycles commute, and any permutation may be written as an (essentially unique) product of disjoint cycles. Since cycles of length 1 have no effect, we can always assume that all numbers from 1 to n are represented in the cycle decomposition of a permutation. If with respect to such a product the number of even-length cycles is odd, then the permutation is *odd* and has *sign* -1; else it is *even* and has sign $+1$. It is easy to show that the sign of permutations determines a homomorphism of S_n into the multiplicative group $\{1,-1\}$. Hence the set of even permutations forms a subgroup A_n of S_n, the *alternating group* on n elements.

Given a permutation σ, for $1 \le i \le n$ let $k_i(\sigma)$ be the number of cycles of length i in the cycle decomposition of σ. For any subgroup Δ of S_n, the *Polya cycle index* $p_\Delta(z_1, \ldots, z_n)$ of Δ is the polynomial

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$$\frac{1}{\#\Delta}\sum_{\sigma\in\Delta}\prod_{i=1}^{n} z_i^{k_i(\sigma)}$$

in indeterminates $z_1, \ldots, z_n$, where $\#\Delta$ is the cardinality (order) of the subgroup Δ. Finally, for any subgroup Δ of S_n, define $\bar{h}(\Delta)$ to be the number of hypergraphs whose automorphism group equals or contains Δ, and let $\bar{d}(\Delta)$ be the number of digraphs whose automorphism group equals or contains Δ. (Note that here digraphs and hypergraphs are labelled structures, i.e., we distinguish between distinct structures even if they are isomorphic.)

Proposition 1. $\bar{h}(\Delta) = 2^{p_\Delta(2,\ldots,2)}$.

Proof. A $\Delta-$*orbit* is a hypergraph of the form $\{\sigma E: \sigma\in\Delta\}$ where E is some subset of $\{1,\ldots, n\}$. Clearly, a hypergraph has automorphism group containing Δ if and only if it is a union of Δ-orbits. Hence it is enough to show that there are exactly $p_\Delta(2,2,\ldots,2)$ Δ-orbits. For any permutation σ let $F(\sigma)$ denote the number of subsets of $\{1,\ldots,n\}$ fixed by σ. By Burnside's Lemma [Lovasz] the number of Δ-orbits is

$$\frac{1}{\#\Delta}\sum_{\sigma\in\Delta}F(\sigma)$$

However, it is easy to see that $F(\sigma) = \prod_{i=1}^{i=n} 2^{k_i(\sigma)}$, since a set S is fixed by σ if and only if it is a union of cycles of σ. Q.E.D.

As shown in [OY], using generating functions given in [Lovasz], for $n\geq 3$, $p_{A_n}(2,\ldots,2) = p_{S_n}(2,\ldots,2)$. In other words, every hypergraph with at least three vertices which is invariant under all even permutations is invariant under all permutations, so we have the following.

Proposition 2. For $n\geq 3$, no hypergraph on n vertices has automorphism group A_n ●

It is easy to construct a digraph whose automorphism group is A_3: namely, a clockwise oriented triangle. However, for $n\geq 4$ the same result holds for digraphs, as we shall see. We shall first give a formula for $\bar{d}(\Delta)$, similar to the formula in Proposition 1. Note that a permutation σ fixes a directed edge (u,v) if and only if it fixes both u and v. Thus $D(\sigma) = k_1(\sigma)(k_1(\sigma)-1)$, where $D(\sigma)$ is the number of directed edges fixed by σ. Again using Burnside's Lemma, the number of Δ-orbits of directed edges is $\frac{1}{\#\Delta}\sum_{\sigma\in\Delta}D(\sigma)$, so reasoning as in Proposition 1 we have:

Proposition 3. $\bar{d}(\Delta) = \frac{1}{\#\Delta}\sum_{\sigma\in\Delta} k_1(\sigma)(k_1(\sigma)-1)$ ●

The formulae established in Propositions 1 and 3 may be used to compute for each subgroup of S_n the number of hypergraphs and digraphs whose automorphism group coincides with Δ [OY]. This can be accomplished using Möbius inversion [Lovasz]. While the method is in principle quite

general, in practice it becomes laborious for $n > 4$. On the other hand, it is relatively easily to show that for $n > 3$ no digraph can have automorphism group A_n, as follows.

Let B_n be the set of odd permutations in S_n. Clearly, it is enough to show that $\bar{d}(A_n) = \bar{d}(S_n)$ if $n > 3$. Since $\#A_n = \#B_n = \frac{1}{2}\#S_n$, it is enough, by Proposition 3, to show that for $n > 3$

$$\sum_{\sigma \in A_n} k_1(\sigma)(k_1(\sigma)-1) = \sum_{\sigma \in B_n} k_1(\sigma)(k_1(\sigma)-1)$$

For $1 \le i \le n$ let $O(n,i)$ (respectively, $E(n,i)$) be the number of odd (respectively, even) permutations σ in S_n such that $k_1(\sigma) = i$. Clearly, $E(n,i) = \binom{n}{i}E(n-i,0)$, with a similar formula holding for the odd permutations, since every permutation in $S(n)$ fixing precisely i vertices can be defined by choosing a set X of size i which it fixes - there are $\binom{n}{i}$ such sets X - and regarding the rest of the permutation, which fixes nothing, as a member of S_{n-i}; note that both permutations are odd or both are even. Thus we can rewrite the above fomula as

$$0 = \sum_{i=1}^{n} \binom{n}{i}(i)(i-1)(O(n-i,0)-E(n-i,0)) \qquad (*)$$

Thus it remains to compute, for each k, the quantity $O(k,0)-E(k,0)$. For each subset Y of $\{1,\ldots,n\}$, write $V(k,Y)$ (respectively, $W(k,Y)$) for the number of odd (respectively, even) permutations in S_k which fix all vertices in Y (and perhaps others besides). Let $D(k,Y) = V(k,Y) - W(k,Y)$. By the principle of inclusion/exclusion [Lovasz], the quantity $O(k,0)-E(k,0)$ may be expressed as the alternating sum

$$\sum_{Y \subseteq \{1,\ldots,k\}} (-1)^{\#Y} D(k,Y)$$

(note that this sum is over all subsets Y, including the empty set.) Clearly, if $\#Y \le k-2$ then $D(k,Y)$ is zero, since any even permutation fixing all vertices in Y may be multiplied by a transposition not affecting Y, and vice-versa. For $\#Y = k-1$ or k, there is only one permutation fixing every vertex in Y - namely, the identity permutation - so $D(k,Y) = -1$. Since there are k sets of size $k-1$ and one of size k, the above formula reduces to $(-1)k(-1)^{k-1}+(-1)(-1)^k$, i.e., $(-1)^k(k-1)$. Substituting into formula (*), where we write $n-i$ for k, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} i(i-1)(-1)^{n-i}(n-i-1)$$

This formula can be rewritten, using some formal differentiation, as

$$aP(-1)+bP'(-1)+cP''(-1)+dP'''(-1)$$

where $P(x) = (1+x)^n$, and a, b, c, and d are functions of n only. Thus, for $n \ge 4$, the expression vanishes identically, so we deduce the following

Proposition 4. For $n \geq 4$, no digraph on n vertices has automorphism group A_n ●

A final remark is that it appears that the above techniques can be extended to showing that other classes of formats [HY,OY] also do not have the alternating group A_n as an automorphism group, for sufficiently large n.

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